# A Theoretical Analysis of Smith and Analytical Predictors

A theoretical analysis has been performed to elucidate the relationship between two popular time delay compensation techniques, the Smith predictor and the analytical predictor. In general, use of these two techniques results in different closed-loop responses. It is only for the special case where the process model is perfect and set point changes occur, that the two techniques are equivalent. A modification of the analytical predictor has been proposed that allows the technique to be used with any feedback controller, instead of the special PI controller employed by Moore et al. (1970).

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## SCOPE

Time delays are a common characteristic of many industrial processes due to the transportation delays associated with fluid flow and the time required to complete a composition analysis. The detrimental effects of time delays on closed-loop stability and feedback control are well known. Consequently, there has been considerable incentive to develop special advanced control techniques that provide time delay compensation. Two popular techniques for process control problems are the Smith predictor (Smith, 1957) and the analytical predictor (Moore, 1969; Moore et al., 1970).

Despite the extensive literature on Smith and analytical predictors, none of the previous papers has explored the relationship between these two popular time delay compensation techniques. However, the assertion has been made that the Smith and analytical predictors are equivalent for a first-order plus time delay system (Ogunnaike and Ray, 1979; Ray, 1981).

In this paper the relationship between the Smith and analytical predictors is derived theoretically for a general class of systems. In general, the Smith and analytical predictors result in different closed-loop systems, and consequently in different control system performance. It is shown that the analytical predictor can be expressed in a form that facilitates comparison with the Smith predictor. Finally, the analytical predictor approach is generalized so that it can be used in conjunction with any feedback controller rather than the special form of PI control that has been employed previously.

## CONCLUSIONS AND SIGNIFICANCE

Our theoretical analysis has shown that, in general, the Smith and analytical predictor techniques result in different control system performance. It is only for the special case where the process model is perfect and no unanticipated load disturbances occur that the two techniques produce equivalent results. The theoretical analysis provides a convenient means of comparing the Smith and analytical predictors via the expressions for the predictions in Eqs. 1 and 21 and the corresponding closed-loop expressions. The block diagrams in Figures 1 and 2 also facilitate this comparison.

A generalized analytical predictor (AP) scheme has been proposed that allows the standard AP approach to be used with any feedback controller. This extension removes one of the key limitations of the standard analytical predictor, namely, that a special type of PI controller be employed.

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## Introduction

The Smith predictor is probably the best known and most widely used time delay compensation technique. Since it was proposed in 1957, the Smith predictor has been the subject of numerous theoretical analyses and experimental applications. Several papers have compared the performance of the Smith predictor with conventional PI and PID controllers (Meyer et al., 1976; Ross, 1977; McMillan, 1983). A number of authors have analyzed the sensitivity of the Smith predictor to model mismatch problems (Palmor and Shinnar, 1978, 1981; Palmor, 1980; Herget et al., 1980; Hocken et al., 1982; Horowitz, 1983). Extensions of the Smith predictor approach to multivariable control problems have also been reported (Alevisakis and Seborg, 1974; Ogunnaike and Ray, 1979; Herget and Fraser, 1981; Owens and Raya, 1982; Watanabe et al., 1983; Palmor and Halevi, 1983). Several authors have proposed modifications of the Smith predictor scheme that involved disturbance estimation or the use of observers (Hang and Wong, 1979; Chiang and Durbin, 1980; Hang and Tham, 1982).

In 1970, Moore developed an alternative time delay compensation scheme, the analytical predictor, which was primarily designed for direct digital control. Later, Doss and Moore (1973) compared the performance of the Smith and the analytical predictors using a laboratory-scale temperature control system. Meyer et al. (1979) reported a similar comparison for a pilot-scale distillation column. In view of the difficulty of generalizing the analytical predictor to higher order process models, Doss and Moore (1981) proposed another version of the analytical predictor developed using $z$-transform techniques. Srinivasan and Mellichamp (1983) have provided a theoretical analysis of the analytical predictor with proportional-only control. The optimal predictor scheme of Donoghue (1977) can be interpreted as a multivariable, continuous-time version of the analytical predictor.

## The Smith Predictor

The Smith predictor technique uses a process model to predict future values of the output variable. The control calculations are based on both the predicted values and the current value of the output. A block diagram of a discrete-time version of the Smith predictor is shown in Figure 1. Transfer functions $G_p(s)$ and $G_m(s)$ are assumed to be rational functions in the Laplace transform variable $s$ since time delays, $\theta_p$ and $\theta_m$, appear in separate terms. Note that the digital controller output signal $u_k$ is based on a predicted future output $\hat{y}_{k+N}$, which is calculated from

$$\hat{y}_{k+N} = y^*_{m_k} + (y_k - y_{m_k}) \tag{1}$$

In Eq. 1, $y_{m_k}$ is the output of a model that contains a time delay; $y^*_{m_k}$ is the output of the same model without the time delay; and $y_k$ is the actual process output. Subscript $k$ denotes the $k$th sampling instant. Integer $N$ represents the model time delay $\theta_m$ expressed as a multiple of the sampling period, $T$. Thus

$$\theta_m = NT \tag{2}$$

For the ideal situation, where the model is perfect and no load disturbances occur, $y_{m_k} = y_k$ and the input signal to the controller is $e_k = r_k - y^*_{m_k}$. Thus the control action is based on $y^*_{m_k}$, the output of the undelayed model, rather than the actual process output, $y_k$. For this ideal situation, the process time delay is removed from the characteristic equation and, in principle, the controller gain can be increased and better control will be achieved. However, this theoretical advantage cannot be fully realized in practice due to the detrimental effects of modeling errors.

If a continuous, first-order process model is selected, then the process models without and with the time delays can be expressed as:

$$\tau_m \frac{dy^*_m(t)}{dt} + y^*_m(t) = K_m u(t) \tag{3}$$

$$\tau_m \frac{dy_m(t)}{dt} + y_m(t) = K_m u(t - \theta_m) \tag{4}$$

If the zero-order hold $H(s)$ in Figure 1 is employed, $u(t)$ is a piecewise constant function, and the analytical solutions to Eqs. 3 and 4 are:

$$y^*_{m_k} = K_m(1 - B)u_{k-1} + By^*_{m_{k-1}} \tag{5}$$

$$y_{m_k} = K_m(1 - B)u_{k-N-1} + By_{m_{k-1}} \tag{6}$$

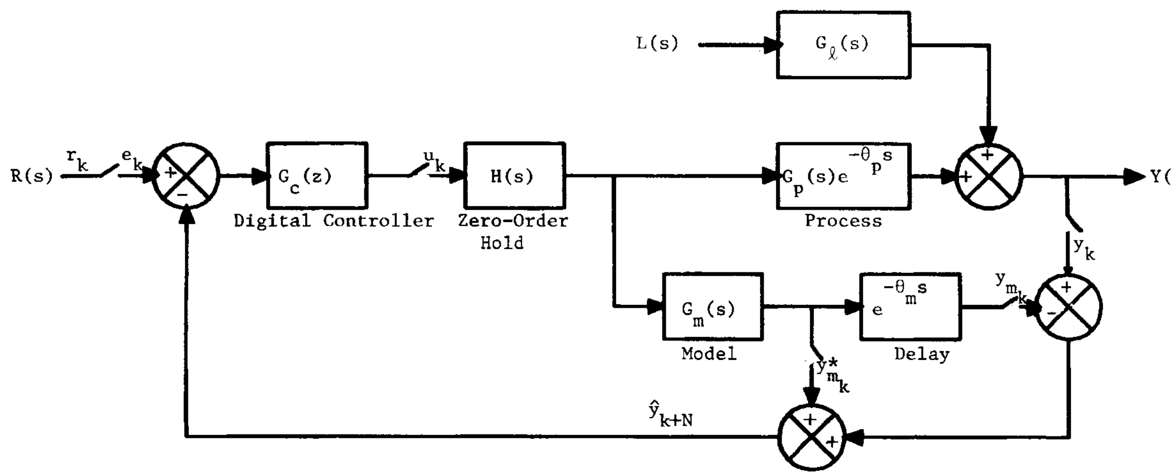


**Figure 1. Block diagram of Smith predictor.**

where $y^*_{m_k}$ and $y_{m_k}$ denote the model outputs at the $k$th sampling instant and $B$ is defined as

$$B = \exp(-T/\tau_m) \tag{7}$$

If $y^*_m$ and $y_m$ have the same initial conditions, it is obvious that

$$y_{m_k} = y^*_{m_{k-N}} \tag{8}$$

For the block diagram in Figure 1, the closed-loop responses to load or set point changes can be obtained from the following expression, which is derived in Appendix A:

$$Y(z) = \frac{G_c(z)G_pH(z)z^{-M}R(z) + [1 + G_c(z)G_mH(z)(1 - z^{-N})]G_\ell L(z)}{1 + G_c(z)[G_mH(z) - G_mH(z)z^{-N} + G_pH(z)z^{-M}]} \tag{9}$$

where the actual process delay $\theta_p$ is assumed to be an integer multiple of the sampling period, $T$, as

$$\theta_p = MT \tag{10}$$

If the assumed model is perfect, then $N = M$, $G_mH(z) = G_pH(z)$, and Eq. 9 reduces to

$$Y(z) = \frac{G_c(z)G_pH(z)z^{-M}R(z) + [1 + G_c(z)G_pH(z)(1 - z^{-M})]G_\ell L(z)}{1 + G_c(z)G_pH(z)} \tag{11}$$

Note that the time delay has been eliminated from the characteristic equation for the ideal situation where the model is perfect.

## The Analytical Predictor

Time delay compensation can also be achieved by predicting the process output one time delay ahead using the current value of the output plus current and past values of the input. Moore (1970) first proposed such an algorithm for direct digital control and called it the analytical predictor. He also included a one-half sampling period correction in the prediction to account for the effect of sampling. We will not consider this feature in our analysis in order to provide a more straightforward comparison between the Smith and analytical predictors.

The analytical predictor was originally derived based on a first-order model. Predictions of future outputs are obtained by solving Eq. 6 recursively and by back-substitution:

$$\hat{y}_{k+1} = By_k + K_m(1 - B)u_{k-N} \tag{12}$$

$$\hat{y}_{k+2} = B\hat{y}_{k+1} + K_m(1 - B)u_{k-N+1} \tag{13}$$

$$\vdots$$

$$\hat{y}_{k+N} = B^N y_k + K_m(1 - B)\sum_{i=1}^{N} B^{i-1}u_{k-i} \tag{14}$$

Expanding Eq. 14 gives

$$\hat{y}_{k+N} = B^N y_k + K_m(1 - B)u_{k-1} + K_m(1 - B)Bu_{k-2} \cdots + K_m(1 - B)B^{N-1}u_{k-N} \tag{15}$$

From Eq. 5, it follows that

$$K_m(1 - B)u_{k-1} = y^*_{m_k} - By^*_{m_{k-1}} \tag{16}$$

$$K_m(1 - B)u_{k-2} = y^*_{m_{k-1}} - By^*_{m_{k-2}} \tag{17}$$

$$\vdots$$

$$K_m(1 - B)u_{k-N} = y^*_{m_{k-N+1}} - By^*_{m_{k-N}} \tag{18}$$

Substituting into Eq. 15 gives

$$\hat{y}_{K+N} = B^N y_k + (y^*_{m_k} - By^*_{m_{k-1}}) + B(y^*_{m_{k-1}} - By^*_{m_{k-2}}) + \cdots + B^{N-1}(y^*_{m_{k-N+1}} - By^*_{m_{k-N}}) \tag{19}$$

or

$$y_{k+N} = B^N y_k + y^*_{m_k} - B^N y^*_{m_{k-N}} \tag{20}$$

But

$$y^*_{m_{k-N}} = y_{m_k}$$

Hence

$$\hat{y}_{k+N} = y^*_{m_k} + B^N(y_k - y_{m_k}) \tag{21}$$

By comparing Eqs. 1 and 21, it is apparent that the Smith predictor and the analytical predictor methods will, in general, furnish different estimates of $\hat{y}_{k+N}$. Since $B^N \neq 1$ in general, these two estimates will be identical only for the special case where the model is perfect and there are no load disturbances. Under these conditions

$$y_{m_k} = y_k \tag{22}$$

and hence

$$\hat{y}_{k+N} = y^*_{m_k} \tag{23}$$

for both predictors. By contrast, for the more realistic conditions where the process model is not perfect and/or load disturbances occur, then the Smith and analytical predictors provide different estimates of $\hat{y}_{k+N}$. They also result in different closed-loop expressions for $Y(z)$, as shown below.

Figure 2 shows a block diagram of the analytical predictor. The corresponding closed-loop expression for $Y(z)$ is

$$Y(z) = \frac{G_c(z)G_pH(z)z^{-M}R(z) + [1 + G_c(z)G_mH(z)(1 - B^N z^{-N})]G_\ell L(z)}{1 + G_c(z)\{G_mH(z) + B^N[G_pH(z)z^{-M} - G_mH(z)z^{-N}]\}} \tag{24}$$

If the model is perfect, Eq. 24 reduces to

$$Y(z) = \frac{G_c(z)G_pH(z)z^{-M}R(z) + [1 + G_c(z)G_pH(z)(1 - B^M z^{-M})]G_\ell L(z)}{1 + G_c(z)G_pH(z)} \tag{25}$$

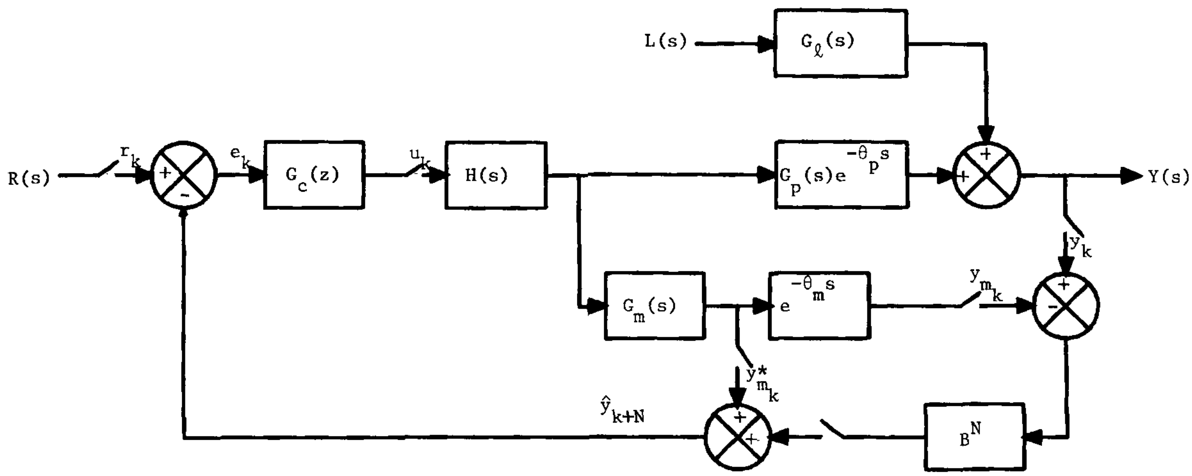


**Figure 2. Block diagram of analytical predictor without load estimation.**

The different performance of the Smith and analytical predictors is apparent from inspection of Eqs. 9, 11, 24, and 25. In general, the Smith and analytical predictors result in significantly different expressions for $Y(z)$. Equations 9 and 24 will not be identical unless the process model is perfect and only set point changes occur, that is $G_pH(z) = G_mH(z)$, $N = M$, and $L(z) = 0$. By contrast, at least two previous publications have stated that the Smith and analytical predictors are identical (Ogunnaike and Ray, 1979; Ray, 1981). Note that when modeling errors are present, use of the Smith and analytical predictors will result in different characteristic equations since the denominators of the transfer functions in Eqs. 9 and 24 are different.

For processes with model error, the difference between the Smith predictor and the analytical predictor is best illustrated by a first-order plus time delay process. Suppose that a digital proportional controller is used and a first-order model is assumed; then the steady state offsets ($\bar{e}$) for a unit step change in the calibrated set point, $r_c = (1 + K_mK_c)r/(K_cK_m)$ for the Smith predictor (SP) and the analytical predictor (AP) are given by

$$(\bar{e})_{SP} = \frac{K_m - K_p}{K_m(1 + K_cK_p)} \tag{26}$$

$$(\bar{e})_{AP} = \frac{(K_m - K_p)[1 + K_cK_m(1 - B^N)]}{K_m[1 + K_cK_m + B^N(K_cK_p - K_cK_m)]} \tag{27}$$

Similarly, for a unit step change in load disturbance, the offsets are

$$(\bar{e})_{SP} = -\frac{K_\ell}{1 + K_cK_p} \tag{28}$$

$$(\bar{e})_{AP} = \frac{-K_\ell[1 + K_cK_m(1 - B)]}{1 + K_cK_m + B^N(K_cK_p - K_cK_m)} \tag{29}$$

These expressions are derived in Appendix A.

It is worth mentioning that there are circumstances where the behavior of the analytical predictor approaches that of the Smith predictor. This situation occurs when the time delay is very small compared to the time constant or when the sampling period is small compared to the time constant. When either of these situations occurs, $B^N$ approaches unity, and hence Eq. 21 for the analytical predictor reduces to Eq. 1 for the Smith predictor regardless of modeling errors or unmeasured load disturbances.

## Analytical Predictor with Load Estimation

One significant disadvantage of the analytical predictor in Figure 2 is that it provides inaccurate steady state predictions of $y_{k+N}$ unless the model is perfect and no load changes occur. Consequently, even if $G_c(z)$ in Figure 2 is a conventional PI or PID controller, offset can result due to biased estimates of $y_{k+N}$. Hammerstrom and Waller (1980) have noted that the same problem occurs for Donoghue's (1977) approach, which is essentially a multivariable, continuous-time version of the analytical predictor.

In order to avoid biased predictions and subsequent offset, Moore modified the analytical predictor by adding a load estimation scheme. Moore's analytical predictor with load estimation has the form:

$$\hat{y}_k = By_{k-1} + K_m(1 - B)(u_{k-N-1} + \hat{d}_{k-1}) \tag{30}$$

$$\hat{d}_k = \hat{d}_{k-1} + K_IT(y_k - \hat{y}_k) \tag{31}$$

$$u_k = K_c(r_{c_k} - \hat{y}_{k+N}) - \hat{d}_k \tag{32}$$

$$\hat{y}_{k+N} = B^Ny_k + K_m(1 - B)\sum_{i=1}^{N} B^{i-1}(u_{k-i} + \hat{d}_k) \tag{33}$$

By comparing with Eqs. 15 and 21, Eq. 33 can be rewritten as

$$\hat{y}_{k+N} = y^*_{m_k} + B^N(y_k - y_{m_k}) + K_m(1 - B)(1 + B + \cdots + B^{N-1})\hat{d}_k$$

or

$$\hat{y}_{k+N} = y^*_{m_k} + B^N(y_k - y_{m_k}) + K_m(1 - B^N)\hat{d}_k \tag{34}$$

To show that this version of the analytical predictor does indeed provide unbiased predictions, rewrite Eq. 30 as

$$\hat{y}_k = By_{k-1} + (y_{m_k} - By_{m_{k-1}}) + K_m(1 - B)\hat{d}_{k-1} \quad (35)$$

Let the steady state values of $y$ and $y_m$ be denoted by $\bar{y}$ and $\bar{y}_m$, respectively. Then from Eq. 31

$$\lim_{k\to\infty} \hat{y}_k = \bar{y} \quad (36)$$

Substituting Eq. 36 into Eq. 35 gives

$$\lim_{k\to\infty} \hat{d}_k = \frac{\bar{y} - \bar{y}_m}{K_m} \quad (37)$$

Hence at steady state, Eq. 34 becomes

$$\lim_{k\to\infty} \hat{y}_{k+N} = \bar{y}_m + B^N(\bar{y} - \bar{y}_m) + (1 - B^N)(\bar{y} - \bar{y}_m) = \bar{y} \quad (38)$$

Equation 38 indicates that the analytical predictor with load estimation gives an accurate steady state prediction regardless of modeling errors or unmeasured load disturbances.

Figure 3 shows a block diagram of Moore's analytical predictor with load estimation. The closed-loop expression for $Y(z)$ is

$$Y(z) = \frac{z^{-M}K_cG_pH(z)R_c(z) + \{1 + G_mH(z)[K_c - K_cA(z)z^{-N} - C(z)z^{-N}]\}\, G_\ell L(z)}{1 + K_cG_mH(z) + [K_cA(z) + C(z)][z^{-M}G_pH(z) - z^{-N}G_mH(z)]} \quad (39)$$

where

$$A(z) = B^N + K_m(1 - B^N)C(z) \quad (40)$$

$$C(z) = \frac{K_IT(1 - Bz^{-1})}{1 - z^{-1}[1 - K_ITK_m(1 - B)]} \quad (41)$$

$$G_mH(z) = \frac{K_m(1 - B)z^{-1}}{1 - Bz^{-1}} \quad (42)$$

For the case of perfect modeling, Eq. 39 reduces to

$$Y(z) = \frac{z^{-M}K_cG_pH(z)R_c(z) + \{1 + G_pH(z)[K_c - K_cA(z)z^{-M} - C(z)z^{-M}]\}G_\ell L(z)}{1 + K_cG_pH(z)} \quad (43)$$

It can be shown that the analytical predictor with load estimation eliminates offset after step changes in either load or set point.

## Generalized Analytical Predictor

Since the analytical predictor with load estimation gives true steady state prediction (see Eq. 38) regardless of the control algorithm, it is not necessary to limit the feedback controller to the special type that Moore proposed. In particular, if the controller and predictor are separated, then any type of feedback control scheme can be used. By contrast, in the AP scheme shown in Figure 3 the control and prediction calculations are coupled via signal $\hat{d}_k$ and tuning parameter $K_I$.

Next, we will show that the controller and predictor can be separated by reformulating the disturbance estimator employed in the AP scheme. Following Moore et al. (1970), assume that the step load disturbance can be estimated after one sampling period when the load transfer function, $G_\ell$, is the same as the process transfer function, $G_p$, and the model is exact. Suppose that a step disturbance of magnitude $\bar{d}$ enters the system at time

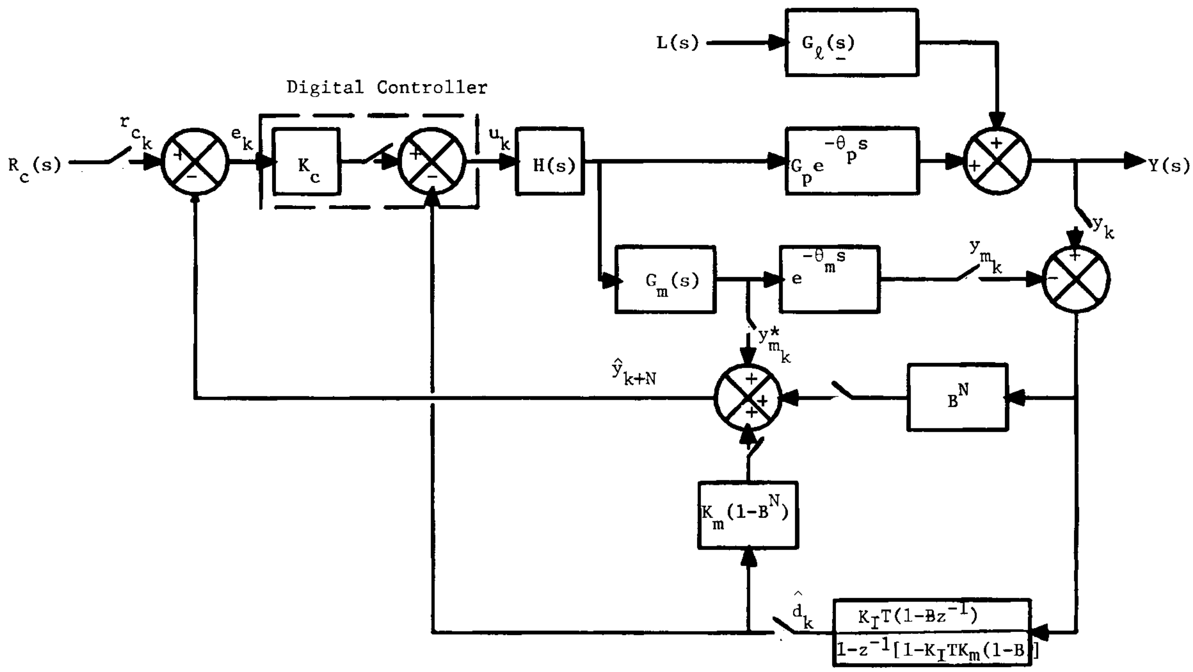


**Figure 3. Block diagram of analytical predictor with load estimation.**

$t = (k - 1)T$; then $\hat{d}_{k-1} = 0$ and from Eq. 30

$$\hat{y}_k = By_{k+1} + K_m(1 - B)u_{k-N-1} \tag{44}$$

If the model is exact and the load transfer function, $G_\ell$, is the same as the process transfer function, $G_p$, then

$$y_k = By_{k-1} + K_m(1 - B)(u_{k-N-1} + \bar{d}) \tag{45}$$

From Eq. 31,

$$\hat{d}_k = K_I T K_m(1 - B)\bar{d} \tag{46}$$

If the step load disturbance is identified after one sampling period, then $\hat{d}_k = \bar{d}$. Hence from Eq. 46,

$$K_I = \frac{1}{TK_m(1 - B)} \tag{47}$$

Substituting Eqs. 30 and 47 into Eq. 31 gives

$$\hat{d}_k = \frac{1}{K_m(1 - B)}[y_k - By_{k-1} - K_m(1 - B)u_{k-N-1}] \tag{48}$$

From Eq. 6

$$K_m(1 - B)u_{k-N-1} = y_{m_k} - By_{m_{k-1}} \tag{49}$$

Hence

$$\hat{d}_k = \frac{1}{K_m(1 - B)}[(y_k - y_{m_k}) - B(y_{k-1} - y_{m_{k-1}})] \tag{50}$$

From Eqs. 34 and 50, the predicted output, $\hat{y}_{k+N}$, becomes

$$\hat{y}_{k+N} = y^*_{m_k} + B^N(y_k - y_{m_k}) + \frac{1 - B^N}{1 - B}[(y_k - y_{m_k}) - B(y_{k-1} - y_{m_{k-1}})] \tag{51}$$

Hence the dependence of the predictor on the controller setting, $K_I$, is eliminated. Moore suggested the same value for $K_I$ (Eq. 47) but did not extend his analysis to derive the general predictor form given by Eq. 51. He also found that using this $K_I$ value and his special PI controller made the closed-loop system very sensitive to modeling errors; consequently, he had to reduce $K_I$ by a factor of 10 to ensure stability (Moore et al., 1970). By contrast, the general predictor given by Eq. 51 can be used with any feedback control scheme that requires an unbiased dead time compensator. We will call this type of dead time compensation scheme the generalized analytical predictor, in order to distinguish it from Moore's original AP scheme. The difference between Moore's analytical predictor with load estimation and the generalized analytical predictor is best illustrated by the block diagrams shown in Figures 3 and 4.

The closed-loop expression for the generalized analytical predictor is

$$Y(z) = \frac{z^{-M}G_c(z)G_pH(z)R(z) + \{1 + G_c(z)G_mH(z)[1 - A^*(z)z^{-N}]\}G_\ell L(z)}{1 + G_c(z)\{G_mH(z) + A^*(z)[z^{-M}G_pH(z) - z^{-N}G_mH(z)]\}} \tag{52}$$

where

$$A^*(z) = B^N + \frac{(1 - B^N)(1 - Bz^{-1})}{1 - B} \tag{53}$$

If the process model is perfect, then

$$Y(z) = \frac{z^{-M}G_c(z)G_pH(z)R(z) + \{1 + G_c(z)G_pH(z)[1 - A^*(z)z^{-M}]\}G_\ell L(z)}{1 + G_c(z)G_pH(z)} \tag{54}$$

Note that the generalized analytical predictor is structurally similar to the Smith predictor. If $A^*(z)$ in Eq. 52 is set equal to unity, Eq. 52 reduces to Eq. 9, which is the closed-loop transfer function for the Smith predictor. One can even say that the Smith predictor is the steady state version of the generalized analytical predictor, since $\lim_{z \to 1} A^*(z) = 1$.

The continuous-time equivalent of the generalized analytical predictor has been found to work well with a suboptimal control scheme and has also been extended to nonlinear and multivariable systems (Wong, 1985).

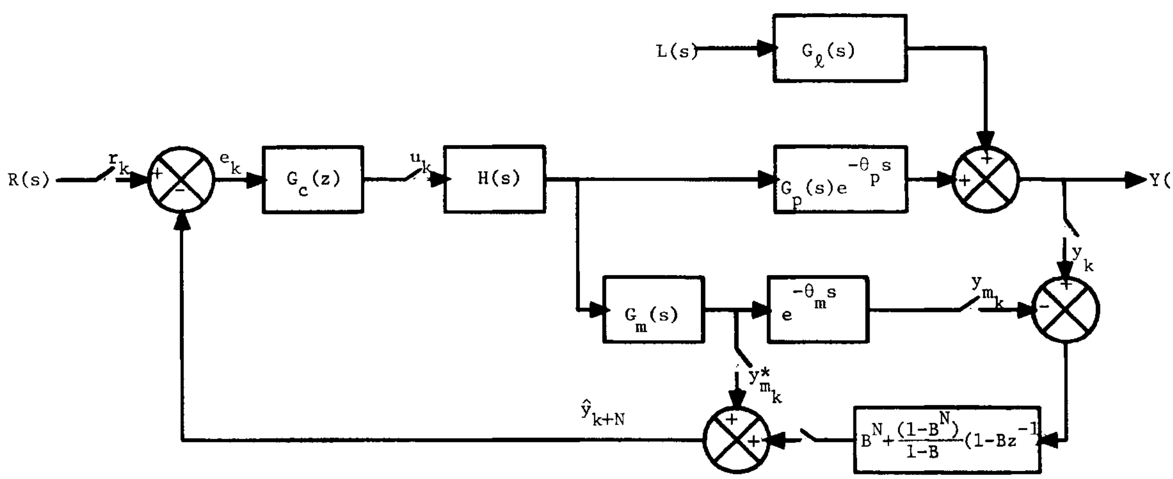


**Figure 4. Block diagram of the generalized analytical predictor.**

## Notation

$A = B^N + K_m(1 - B)C$
$A^* = B^N + [(1 - B^N)(1 - Bz^{-1})/(1 - B)]$
$B = \exp(-T/\tau_m)$
$B' = \exp(-T/\tau_p)$
$C$ = defined in Eq. B5
$\hat{d}$ = estimated load disturbance
$e$ = error signal
$\bar{e}$ = steady state error
$G_c$ = controller transfer function
$G_\ell$ = load transfer function
$G_m$ = model transfer function
$G_p$ = process transfer function
$H$ = zero-order hold
$K_I$ = integral constant for load estimation
$K_\ell$ = load steady state gain
$K_m$ = model steady state gain
$l$ = load disturbance
$L(s)$ = Laplace transform of $l$
$r$ = set point
$r_c$ = calibrated set point
$R(s)$ = Laplace transform of $r$
$T$ = sampling period
$u$ = controller output
$y$ = process output
$y_m$ = delayed model output
$y_m^*$ = undelayed model output
$\hat{y}_{k+N}$ = predicted process output $N$ sampling periods ahead
$\bar{y}$ = steady state process output
$\bar{y}_m$ = steady state model output
$Y_p(z)$ = transform of $\hat{y}_{k+N}$

### Greek letters

$\theta_m$ = model time delay
$\theta_p$ = process time delay
$\tau_m$ = model time constant
$\tau_p$ = process time constant

## Appendix A. Closed Loop Relationships for Smith and Analytical Predictors

### *Smith predictor*

Let $Y_p(z)$ be the $z$-transform of $\hat{y}_{k+N}$. From the block diagram, Figure 1,

$$U(z) = G_c(z)[R(z) - Y_p(z) = \frac{G_c(z)[R(z) - Y(z)]}{1 + G_c(z)G_mH(z)(1 - z^{-N})} \tag{A1}$$

Also

$$Y(z) = z^{-M}G_pH(z)U(z + G_\ell L(z) = \frac{z^{-M}G_pH(z)G_c(z)[R(z) - Y(z)]}{1 + G_c(z)G_mH(z)(1 - z^{-N})} + G_\ell L(z) \tag{A2}$$

Rearranging gives

$$Y(z) = \frac{G_c(z)G_pH(z)z^{-M}R(z) + [1 + G_c(z)G_mH(z)(1 - z^{-N})]G_\ell L(z)}{1 + G_c(z)[G_mH(z) - G_mH(z)z^{-N} + G_pH(z)z^{-M}]} \tag{A3}$$

For a digital proportional controller and a first order process (and model)

$$G_c(z) = K_c \tag{A4}$$

$$G_pH(z) = \frac{K_p(1 - B')z^{-1}}{1 - B'z^{-1}} \tag{A5}$$

$$G_mH(z) = \frac{K_m(1 - B)z^{-1}}{1 - Bz^{-1}} \tag{A6}$$

where

$$B = e^{-T/\tau_m} \tag{A7}$$

$$B' = e^{-T/\tau_p} \tag{A8}$$

For a unit step change in set point with set point calibration employed

$$R(z) = \frac{1 + K_cK_m}{K_cK_m(1 - z^{-1})} \tag{A9}$$

From the final value theorem

$$\bar{y} = \lim_{k\to\infty} y_k = \lim_{z\to 1}(1 - z^{-1}Y(z) = \lim_{z\to 1} \frac{\dfrac{(1 + K_cK_mK_cK_p(1 - B')z^{-M-1}}{K_cK_m(1 - B'z^{-1})}}{1 + K_c\left[\dfrac{K_m(1 - B)z^{-1}}{1 - Bz^{-1}} - \dfrac{K_m(1 - B)z^{-N-1}}{1 - Bz^{-1}} + \dfrac{K_p(1 - B')z^{-M-1}}{1 - B'z^{-1}}\right]} = \frac{K_p(1 + K_cK_m)}{K_m(1 + K_cK_p)} \tag{A10}$$

Hence the offset $\bar{e}$ is

$$\bar{e} = 1 - \bar{y} = \frac{K_m - K_p}{K_m(1 + K_cK_p)} \tag{A11}$$

Hence, the system exhibits offset unless $K_m = K_p$.

Similarly for a unit step change in load

$$\bar{y} = \lim_{k\to\infty} y_k = \frac{K_\ell}{1 + K_cK_p} \tag{A12}$$

and

$$\bar{e} = -\frac{K_\ell}{1 + K_cK_p} \tag{A13}$$

### *Analytical Predictor*

From Figure 2

$$U(z) = G_c(z)[R(z) - Y_p(z)] = \frac{G_c(z)[R(z) - B_*^NY(z)]}{1 + G_c(z)G_mH(z)(1 - B^Nz^{-N})} \tag{A14}$$

Also

$$Y(z) = z^{-M}G_pH(z)U(z) + G_\ell L(z)$$
$$= \frac{z^{-M}G_pH(z)G_c(z)[R(z) - B^NY(z)]}{1 + G_c(z)G_mH(z)(1 - B^Nz^{-N})} + G_\ell L(z) \quad \text{(A15)}$$

Rearranging gives

$$Y(z) = \frac{G_c(z)G_pH(z)z^{-M}R(z) + [1 + G_c(z)G_mH(z)(1 - B^Nz^{-N})]G_\ell L(z)}{1 + G_c(z)\{G_mH(z) + B^N[G_pH(z)z^{-M} - G_mH(z)z^{-N}]\}} \quad \text{(A16)}$$

For a digital proportional controller and a first-order process (and model), a unit step change in set point gives

$$\bar{y} = \lim_{k\to\infty} y_k$$
$$= \lim_{z\to 1} \frac{\dfrac{(1 + K_cK_m)K_cK_p(1 - B')z^{-M-1}}{K_cK_m(1 - B'z^{-1})}}{1 + K_c\left\{\dfrac{K_m(1 - B)z^{-1}}{1 - Bz^{-1}} + B^N\left[\dfrac{K_p(1 - B')z^{-M-1}}{1 - B'z^{-1}} - \dfrac{K_m(1 - B)z^{-N-1}}{1 - Bz^{-1}}\right]\right\}}$$

or

$$\bar{y} = \frac{K_p(1 + K_cK_m)}{K_m[1 + K_cK_m + B^N(K_cK_p - K_cK_m)]} \quad \text{(A17)}$$

Hence the offset is

$$\bar{e} = \frac{(K_m - K_p)[1 + K_cK_m(1 - B^N)]}{K_m(1 + K_cK_m + B^N(K_cK_p - K_cK_m)]} \quad \text{(A18)}$$

Thus there is an offset unless $K_m = K_p$.

Similarly for a unit step change in load

$$\bar{y} = \lim_{k\to\infty} y_k = \frac{K_\ell[1 + K_cK_m(1 - B^N)]}{1 + K_c[K_m + B^N(K_p - K_m)]} \quad \text{(A19)}$$

and

$$\bar{e} = -\frac{K_\ell[1 + K_cK_m(1 - B^N)]}{1 + K_c[K_m + B^N(K_p - K_m)]} \quad \text{(A20)}$$

## Appendix B. Closed-Loop Relationships for the Analytical Predictor with Load Estimation

### *Moore's scheme*

Moore's controller has the form

$$u_k = K_c(r_{c_k} - \hat{y}_{k+N}) - \hat{d}_k \quad \text{(B1)}$$

where

$$r_{c_k} = \frac{1 + K_cK_m}{K_cK_m} r_k \quad \text{(B2)}$$

From Eqs. 3 and 31

$$\hat{d}_k = [1 - K_ITK_m(1 - B)]\hat{d}_{k-1} + K_IT[(y_k - y_{m_k}) - B(y_{k-1} - y_{m_{k-1}})] \quad \text{(B3)}$$

Taking the $z$-transform gives

$$D(z) = C(z)[Y(z) - Y_m(z)] \quad \text{(B4)}$$

where

$$C(z) = \frac{K_IT(1 - Bz^{-1})}{1 - z^{-1}[1 - K_ITK_m(1 - B)]} \quad \text{(B5)}$$

From Eq. 33

$$Y_p(z) = Y_m^*(z) + B^N[Y(z) - Y_m(z)] + K_m(1 - B^N)D(z) \quad \text{(B6)}$$

Hence

$$Y_p(z) = Y_m^*(z) + A(z)[Y(z) - Y_m(z)] \quad \text{[B7]}$$

where

$$A(z) = B^N + K_m(1 - B^N)C(z) \quad \text{(B8)}$$

Taking the $z$-transform of Eq. B1 yields

$$U(z) = K_c[R_c(z) - Y_p(z)] - D(z) \quad \text{(B9)}$$

or

$$U(z) = \frac{K_cR_c(z) - [K_cA(z) + C(z)]Y(z)}{1 + G_mH(z)[K_c - K_cA(z)z^{-N} - C(z)z^{-N}]} \quad \text{(B10)}$$

But

$$Y(z) = z^{-M}G_pH(z)U(z) + G_\ell L(z) \quad \text{(B11)}$$

Solving gives

$$Y(z) = \frac{z^{-M}K_cG_pH(z)R_c(z) + \{1 + G_mH(z)[K_c - K_cA(z)_z^{-N} - C(z)_z^{-N}]\}G_\ell L(z)}{1 + K_cG_mH(z) + [K_cA(z) + C(z)][z^{-M}G_pH(z) - z^{-N}G_mH(z)]} \quad \text{(B12)}$$

For a perfect model, Eq. B12 reduces to

$$Y(z) = \frac{z^{-M}K_cG_pH(z)R_c(z) + \{1 + G_pH(z)[K_c - K_cA(z)z^{-M} - C(z)z^{-M}]\}G_\ell L(z)}{1 + K_cG_pH(z)} \tag{B13}$$

### *Generalized analytical predictor*

Substituting the expression given by Eq. 47, Eq. B4 becomes

$$D(z) = \frac{(1 - Bz^{-1})[Y(z) - Y_m(z)]}{K_m(1 - B)} \tag{B14}$$

Similarly

$$Y_p(z) = Y_m^*(z) + A^*(z)[Y(z) - Y_m(z)] \tag{B15}$$

where

$$A^*(z) = B^N + \frac{(1 - B)(1 - Bz^{-1})}{1 - B} \tag{B16}$$

But

$$U(z) = G_c(z)[R(z) - Y_p(z)] \tag{B17}$$

Hence

$$U(z) = \frac{G_c(z)[R(z) - A^*(z)Y(z)]}{1 + G_c(z)G_mH(z)[1 - A^*(z)z^{-N}]} \tag{B18}$$

Substituting Eq. B18 into Eq. B11 gives

$$Y(z) = \frac{z^{-M}G_c(z)G_pH(z)R(z) + \{1 + G_c(z)G_mH(z)[1 - A^*(z)z^{-N}]\}G_LL(z)}{1 + G_c(z)\{G_mY(z) + A^*(z)[z^{-M}G_pH(z) - z^{-N}G_mH(z)]\}} \tag{B19}$$

For a perfect model, this reduces to

$$Y(z) = \frac{z^{-M}G_c(z)G_pH(z)R(z) + \{1 + G_c(z)G_pH(z)[1 - A^*(z)z^{-M}]\}G_\ell L(z)}{1 + G_c(z)G_pH(z)} \tag{B20}$$

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